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Vorticity-preserving schemes for the compressible Euler equations

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Abstract

The concept of vorticity-preserving scheme introduced by Morton and Roe is considered for the system wave equation and extended to the linearised and full compressible Euler equations. Useful criteria are found for a general dissipative conservative scheme to be vorticity preserving. Using them, a residual-based scheme is shown to be vorticity preserving for the Euler equations, which is confirmed by vortex flow calculations.

Keywords: Vorticity preserving; Dissipative scheme; Euler equations; Vortex advection

1. Introduction

Reducing the numerical diffusion of vortices is a key point in many computational fluid dynamics problems such as the simulation of aircraft trailing vortices, blade–vortex interaction of helicopter rotors, rotor–stator interaction of turbo-shaft engines, some aeroacoustic problems and weather forecasting. A classical approach for reducing the numerical diffusion of vorticity is to enhance the order of accuracy of the discretisation scheme, either by increasing the number of points in each space direction (as in [1–3] for instance) or in a more compact way (e.g. [4–9]).

Another point of view has been proposed by Morton and Roe [10] through the concept of vorticity preserving in pure acoustics. In the acoustic model, any vorticity field is time invariant and Morton and Roe have found that the Lax–Wendroff–Ni scheme [11], also known as the rotated Richtmyer scheme, preserves exactly a discrete analogue of the vorticity. In the present paper, we pursue this investigation and try to characterise general dissipative schemes preserving vorticity. Our starting point is a continuous initial-value problem for the system wave equation (acoustics) augmented with a general dissipative term, for which a criterion is found

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for the dissipation to have no effect on the vorticity field. Then the study is transposed at the discrete level to handle conservative schemes of arbitrary order of accuracy. After that, the definition of vorticity preserving is extended to the linearised Euler equations describing acoustics with advection. Useful conditions can be derived for a dissipative scheme to approximate the vorticity transport equation without any numerical diffusion. Finally, the analysis is suitably extended to the full Euler equations to provide a similar result.

As a first consequence, the present work proves that a residual-based scheme of second-order accuracy, named RBV2, is vorticity preserving for the Euler equations. To check this property, numerical applications are presented for steady and unsteady vortex flows. The RBV2 scheme is also compared to a classical scheme of third-order accuracy in space.

2. Pure acoustics

2.1. Continuous analysis

Pure acoustics is governed by the Euler equations linearised around a flow at rest. In a dimensionless form where the sound speed and density of the flow at rest are equal to one and in two space dimensions, these equations can be written as:

$$\partial_t w + \partial_x f + \partial_y g = 0 \tag{1}$$

with

$$w = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad f = f(w) = \begin{bmatrix} u \\ p \\ 0 \end{bmatrix}, \quad g = g(w) = \begin{bmatrix} v \\ 0 \\ p \end{bmatrix}, \tag{2}$$

where t is the time, x and y are Cartesian space coordinates, u and v denote the x and y-components of the perturbed fluid velocity $\vec{V}$ and p is the perturbed pressure. The perturbed density is equal to p. Of course, we could replace the hyperbolic system (1) by the classical wave equation

$$\partial_{tt} p = \partial_{xx} p + \partial_{yy} p,$$

but we prefer to keep the system form for further extensions to the linearised and full Euler equations.

The velocity equations, i.e. the two last equations of the system, express

$$\partial_t \vec{V} + \vec{\nabla} p = 0, \tag{3}$$

so that by taking the curl of (3), the pressure disappears and we get

$$\partial_t \omega = 0, \tag{4}$$

where $\omega = \partial_x v - \partial_y u$ is the vorticity. More precisely in the present 2-D case, the vorticity vector $\vec{\omega} = \vec{\nabla} \times \vec{V}$ is normal to the x, y-plane and ω is the normal component. Eq. (4) shows that in pure acoustics, the vorticity is preserved in time.

At the discrete level, it is easy to find centred discretisations of the pressure gradient and of the vorticity that lead to a similar vorticity preservation. The difficulty arises when numerical dissipation (either added or induced by upwinding) is taken into account. The question is thus to characterise dissipative schemes preserving vorticity. To address this issue, we first consider a continuous model of dissipative scheme. Let us begin with the simple model:

$$\partial_t w + \partial_x f + \partial_y g = \varepsilon(\partial_{xx} w + \partial_{yy} w), \tag{5}$$

where w, f and g are given by (2) and ε is a positive parameter. By taking the curl of the velocity equations of (5), we obtain

$$\partial_t \omega = \varepsilon(\partial_{xx}\omega + \partial_{yy}\omega) \tag{6}$$

and vorticity preservation is lost. Instead, vorticity is diffused by the Laplacian operator in the right-hand side of (6).

To go beyond, we consider the more general dissipative system:

$$\partial_t w + \partial_x f + \partial_y g = \varepsilon_1 \partial_x(\Phi_1 q) + \varepsilon_2 \partial_y(\Phi_2 q), \tag{7}$$

where w, f and g are given by (2), ε_1 and ε_2 are positive parameters, Φ_1 (respectively Φ_2) is a constant matrix having the same eigenvectors as the Jacobian matrix $A = \mathrm{d}f/\mathrm{d}w$ (resp. $B = \mathrm{d}g/\mathrm{d}w$) and q is a vector involving odd derivatives in space or time of w, f and g.

Theorem 1. *The acoustic system with dissipation (7) is vorticity preserving for any choice of q if and only if*

$$\varepsilon_1 \Phi_1 = \varepsilon A, \quad \varepsilon_2 \Phi_2 = \varepsilon B, \tag{8}$$

where ε is a scalar parameter.

Proof 1. For acoustics, the eigenvalues of A and B are

$$a^{(1)} = b^{(1)} = -1, \quad a^{(2)} = b^{(2)} = 0, \quad a^{(3)} = b^{(3)} = 1$$

and we have

$$A = T_A \,\mathrm{Diag}[a^{(i)}] T_A^{-1}, \quad B = T_B \,\mathrm{Diag}[b^{(i)}] T_B^{-1},$$

with the unitary matrices

$$T_A = \begin{bmatrix} -\gamma & 0 & \gamma \\ \gamma & 0 & \gamma \\ 0 & 1 & 0 \end{bmatrix}, \quad T_B = \begin{bmatrix} -\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & \gamma \end{bmatrix}, \quad T_A^{-1} = T_A^t, \quad T_B^{-1} = T_B^t,$$

where $\gamma = 1/\sqrt{2}$.

The matrices Φ_p can be expressed in terms of their eigenvalues $\varphi_p^{(i)}$ as:

$$\Phi_1 = T_A \,\mathrm{Diag}[\varphi_1^{(i)}] T_A^{-1} = \begin{bmatrix} \frac{1}{2}(\varphi_1^{(3)} + \varphi_1^{(1)}) & \frac{1}{2}(\varphi_1^{(3)} - \varphi_1^{(1)}) & 0 \\ \frac{1}{2}(\varphi_1^{(3)} - \varphi_1^{(1)}) & \frac{1}{2}(\varphi_1^{(3)} + \varphi_1^{(1)}) & 0 \\ 0 & 0 & \varphi_1^{(2)} \end{bmatrix},$$

$$\Phi_2 = T_B \,\mathrm{Diag}[\varphi_2^{(i)}] T_B^{-1} = \begin{bmatrix} \frac{1}{2}(\varphi_2^{(3)} + \varphi_2^{(1)}) & 0 & \frac{1}{2}(\varphi_2^{(3)} - \varphi_2^{(1)}) \\ 0 & \varphi_2^{(2)} & 0 \\ \frac{1}{2}(\varphi_2^{(3)} - \varphi_2^{(1)}) & 0 & \frac{1}{2}(\varphi_2^{(3)} + \varphi_2^{(1)}) \end{bmatrix}.$$

By taking the curl of the velocity equations of (7), we obtain the vorticity equation:

$$\partial_t \omega = \varepsilon_1 \partial_x curl_1 + \varepsilon_2 \partial_y curl_2 \tag{9}$$

with

$$curl_p = \partial_x(\Phi_p q)^{(3)} - \partial_y(\Phi_p q)^{(2)}, \quad p = 1, 2,$$

where the superscript refer to the vector components.

Using the above expression of Φ_1 and Φ_2, we get

$$curl_1 = \varphi_1^{(2)} \partial_x q^{(3)} - \frac{1}{2}(\varphi_1^{(3)} - \varphi_1^{(1)}) \partial_y q^{(1)} - \frac{1}{2}(\varphi_1^{(3)} + \varphi_1^{(1)}) \partial_y q^{(2)},$$

$$curl_2 = \frac{1}{2}(\varphi_2^{(3)} - \varphi_2^{(1)}) \partial_x q^{(1)} + \frac{1}{2}(\varphi_2^{(3)} + \varphi_2^{(1)}) \partial_x q^{(3)} - \varphi_2^{(2)} \partial_y q^{(2)}.$$

Denoting $\bar{\varphi}_p^{(i)} = \varepsilon_p \varphi_p^{(i)}$, the right-hand side of Eq. (9) becomes

$$\mathrm{RHS} = \bar{\varphi}_1^{(2)} \partial_{xx} q^{(3)} + \frac{1}{2}(\bar{\varphi}_2^{(3)} - \bar{\varphi}_2^{(1)} - \bar{\varphi}_1^{(3)} + \bar{\varphi}_1^{(1)}) \partial_{xy} q^{(1)} - \frac{1}{2}(\bar{\varphi}_1^{(3)} + \bar{\varphi}_1^{(1)}) \partial_{xy} q^{(2)}$$
$$+ \frac{1}{2}(\bar{\varphi}_2^{(3)} + \bar{\varphi}_2^{(1)}) \partial_{xy} q^{(3)} - \bar{\varphi}_2^{(2)} \partial_{yy} q^{(2)}.$$

The necessary and sufficient condition for (7) to be vorticity preserving, i.e. for *RHS* to be zero, for any $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$ is

$$\begin{cases} \bar{\varphi}_1^{(2)} = 0, \quad \bar{\varphi}_2^{(2)} = 0, \\ \bar{\varphi}_1^{(3)} = -\bar{\varphi}_1^{(1)}, \quad \bar{\varphi}_2^{(3)} = -\bar{\varphi}_2^{(1)}, \\ \bar{\varphi}_1^{(3)} - \bar{\varphi}_1^{(1)} = \bar{\varphi}_2^{(3)} - \bar{\varphi}_2^{(1)} \end{cases}$$

which gives

$$\bar{\varphi}_1^{(1)} = \bar{\varphi}_2^{(1)} = -\varepsilon, \quad \bar{\varphi}_1^{(2)} = \bar{\varphi}_2^{(2)} = 0, \quad \bar{\varphi}_1^{(3)} = \bar{\varphi}_2^{(3)} = \varepsilon, \tag{10}$$

where ε is an arbitrary parameter. So the eigenvalues of $\varepsilon_1 \Phi_1$ (resp. $\varepsilon_2 \Phi_2$) are proportional to those of A (resp. B). Since the eigenvectors of Φ_1 (resp. Φ_2) are those of A (resp. B), we obtain the condition (8). □

Example: The dissipation operator in the Lax–Wendroff method is defined by

$$\Phi_1 = A, \quad \Phi_2 = B, \quad \varepsilon_1 = \varepsilon_2 = \frac{\Delta t}{2},$$
$$q = \partial_x f + \partial_y g.$$

With these choices, System (7) becomes:

$$\partial_t w + \partial_x f + \partial_y g = \frac{\Delta t}{2} [\partial_x (A^2 \partial_x w + AB \partial_y w) + \partial_y (BA \partial_x w + B^2 \partial_y w)]. \tag{11}$$

Since Condition (8) holds with $\varepsilon = \Delta t/2$, this system is vorticity preserving. Here we recover at the continuous level the result found by Morton and Roe [10].

2.2. Discrete analysis

To transfer at the discrete level the above result on vorticity preservation, we only need a single difference approximation of the x and y-derivatives involved in System (7) and the same discretisation of q in its two occurrences. Consider the general conservative approximation of System (1):

$$D_0 \mu_0 w + D_1 \mu f + D_2 \mu g = h_1 D_1 (\Phi_1 \tilde{q}) + h_2 D_2 (\Phi_2 \tilde{q}), \tag{12}$$

where D_0, D_1 and D_2 are difference operators, respectively consistent with the t, x and y-derivatives, μ_0 and μ are discrete spatial operators consistent with the identity that may represent some averaging, h_1 and h_2 are positive parameters depending on powers of the steps Δt, δx, δy and tending to zero with them, Φ_1 (resp. Φ_2) is a matrix having the same eigenvectors as $A = \mathrm{d}f/\mathrm{d}w$ (resp. $B = \mathrm{d}g/\mathrm{d}w$) and $\tilde{q}$ is a vector involving discrete functions of w, f and g.

An important point is that the space operators D_1, D_2, μ and μ_0 are *centred* and the right-hand side of (12) is supposed to be a numerical dissipation. The accuracy of the space operators may be of any even order.

The discrete vorticity is defined as

$$\tilde{\omega} = D_1 v - D_2 u, \tag{13}$$

where u and v are the fluid velocity components.

For pure acoustics, vorticity preservation means

$$D_0 \mu_0 \tilde{\omega} = 0. \tag{14}$$

Theorem 2. *For the acoustic system, the scheme* (12) *is vorticity preserving for any choice of* $\tilde{q}$ *if and only if*

$$h_1 \Phi_1 = hA, \quad h_2 \Phi_2 = hB, \tag{15}$$

where h *is a scalar parameter.*

Proof 2. By taking the discrete curl of the velocity equations of the scheme (12), we obtain the discrete vorticity equation:

$$D_0\mu_0\tilde{curl}_w + D_1\mu\tilde{curl}_f + D_2\mu\tilde{curl}_g = h_1D_1\tilde{curl}_1 + h_2D_2\tilde{curl}_2, \tag{16}$$

where

$$\begin{aligned}
&\tilde{curl}_w = D_1w^{(3)} - D_2w^{(2)},\\
&\tilde{curl}_f = D_1f^{(3)} - D_2f^{(2)}, \quad \tilde{curl}_g = D_1g^{(3)} - D_2g^{(2)},\\
&\tilde{curl}_p = D_1(\Phi_p\tilde{q})^{(3)} - D_2(\Phi_p\tilde{q})^{(2)}, \quad p = 1,2.
\end{aligned}$$

Calculations of the terms in the left-hand side of Eq. (16) gives

$$\tilde{\mathrm{LHS}} = D_0\mu_0(D_1v - D_2u) - D_1\mu D_2p + D_2\mu D_1p = D_0\mu_0\tilde{\omega}.$$

Similarly as in Proof 1, by denoting $\bar{\varphi}_p^{(i)} = h_p\varphi_p^{(i)}$, the right-hand side of Eq. (16) can be written as

$$\begin{aligned}
\tilde{\mathrm{RHS}} = {}&\bar{\varphi}_1^{(2)}D_1D_1\tilde{q}^{(3)} + \frac{1}{2}(\bar{\varphi}_2^{(3)} - \bar{\varphi}_2^{(1)} - \bar{\varphi}_1^{(3)} + \bar{\varphi}_1^{(1)})D_1D_2\tilde{q}^{(1)} - \frac{1}{2}(\bar{\varphi}_1^{(3)} + \bar{\varphi}_1^{(1)})D_1D_2\tilde{q}^{(2)}\\
&+\frac{1}{2}(\bar{\varphi}_2^{(3)} + \bar{\varphi}_2^{(1)})D_1D_2\tilde{q}^{(3)} - \bar{\varphi}_2^{(2)}D_2D_2\tilde{q}^{(2)}.
\end{aligned} \tag{17}$$

The necessary and sufficient condition for $\tilde{\mathrm{RHS}}$ to be zero for any $\tilde{q}^{(1)}$, $\tilde{q}^{(2)}$ and $\tilde{q}^{(3)}$ is

$$\bar{\varphi}_1^{(1)} = \bar{\varphi}_2^{(1)} = -h, \quad \bar{\varphi}_1^{(2)} = \bar{\varphi}_2^{(2)} = 0, \quad \bar{\varphi}_1^{(3)} = \bar{\varphi}_2^{(3)} = h, \tag{18}$$

where h is an arbitrary parameter, which matches the condition (15). □

Examples: Consider a Cartesian mesh ($x_j = j\delta x$, $y_k = k\delta y$) and define the difference and average operators:

$$\begin{aligned}
&(\delta_1v)_{j+\frac{1}{2},k} = v_{j+1,k} - v_{j,k}, \quad (\mu_1v)_{j+\frac{1}{2},k} = \frac{1}{2}(v_{j+1,k} + v_{j,k}),\\
&(\delta_2v)_{j,k+\frac{1}{2}} = v_{j,k+1} - v_{j,k}, \quad (\mu_2v)_{j,k+\frac{1}{2}} = \frac{1}{2}(v_{j,k+1} + v_{j,k}),
\end{aligned}$$

where $2j$ and $2k$ are integers, so that for instance:

$$(\delta_1v)_{j,k} = v_{j+\frac{1}{2},k} - v_{j-\frac{1}{2},k}, \quad (\mu_1v)_{j,k} = \frac{1}{2}(v_{j+\frac{1}{2},k} + v_{j-\frac{1}{2},k}).$$

For the time discretisation $t^n = n\Delta t$, we use the time difference operators:

$$(\Delta v)^n = v^{n+1} - v^n, \tag{19}$$

$$(\overline{\Delta}v)^{n+1} = \frac{3}{2}v^{n+1} - 2v^n + \frac{1}{2}v^{n-1}, \tag{20}$$

where n is a positive integer.

(a) Consider the Lax–Wendroff–Ni (LWN) scheme [11]:

$$\frac{\Delta w}{\Delta t} + \frac{\delta_1\mu_1\mu_2^2}{\delta x}f + \frac{\delta_2\mu_2\mu_1^2}{\delta y}g = \frac{\Delta t}{2}\frac{\delta_1\mu_2}{\delta x}\left[A\left(\frac{\delta_1\mu_2}{\delta x}f + \frac{\delta_2\mu_1}{\delta y}g\right)\right] + \frac{\Delta t}{2}\frac{\delta_2\mu_1}{\delta y}\left[B\left(\frac{\delta_1\mu_2}{\delta x}f + \frac{\delta_2\mu_1}{\delta y}g\right)\right]. \tag{21}$$

This scheme is of the form (12) with

$$\begin{aligned}
&D_0 = \frac{1}{\Delta t}\Delta, \quad D_1 = \frac{1}{\delta x}\delta_1\mu_2, \quad D_2 = \frac{1}{\delta y}\delta_2\mu_1, \quad \mu_0 = \mathrm{I}, \quad \mu = \mu_1\mu_2\\
&\Phi_1 = A, \quad \Phi_2 = B, \quad h_1 = h_2 = \frac{\Delta t}{2}, \quad \tilde{q} = \frac{\delta_1\mu_2}{\delta x}f + \frac{\delta_2\mu_1}{\delta y}g.
\end{aligned}$$

The discrete vorticity (13) is the compact vorticity:

$$\tilde{\omega}_{j+\frac{1}{2},k+\frac{1}{2}} = \left(\frac{\delta_1\mu_2}{\delta x}v - \frac{\delta_2\mu_1}{\delta y}u\right)_{j+\frac{1}{2},k+\frac{1}{2}}. \tag{22}$$

Condition (15) is satisfied with $h = \Delta t/2$, so that the LWN scheme preserves the discrete vorticity, namely

$$\left(\frac{\Delta\tilde{\omega}}{\Delta t}\right)^n_{j+\frac{1}{2},k+\frac{1}{2}} = 0,$$

or else

$$\tilde{\omega}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \tilde{\omega}^n_{j+\frac{1}{2},k+\frac{1}{2}} \tag{23}$$

as already found by Morton and Roe [10].

(b) Consider a slightly different version of the Lax–Wendroff scheme defined on the same 3×3-point stencil as

$$\frac{\Delta w}{\Delta t} + \frac{\delta_1\mu_1}{\delta x}f + \frac{\delta_2\mu_2}{\delta y}g = \frac{\Delta t}{2}\frac{\delta_1}{\delta x}\left[A\left(\frac{\delta_1}{\delta x}f + \frac{\delta_2\mu_1\mu_2}{\delta y}g\right)\right] + \frac{\Delta t}{2}\frac{\delta_2}{\delta y}\left[B\left(\frac{\delta_1\mu_1\mu_2}{\delta x}f + \frac{\delta_2}{\delta y}g\right)\right]. \tag{24}$$

This scheme cannot be put in the form (12) because there is not a single $\tilde{q}$ on the right-hand side of Eq. (24). Scheme (24) is of the form:

$$D_0\mu_0 w + D_1\mu f + D_2\mu g = h_1 D_1(\Phi_1\tilde{q}_1) + h_2 D_2(\Phi_2\tilde{q}_2) \tag{25}$$

with

$$D_0 = \frac{1}{\Delta t}\Delta, \quad D_1 = \frac{1}{\delta x}\delta_1\mu_1, \quad D_2 = \frac{1}{\delta y}\delta_2\mu_2, \quad \mu_0 = \mu = \mathrm{I},$$

$$\Phi_1 = A, \quad \Phi_2 = B, \quad h_1 = h_2 = \frac{\Delta t}{2}, \quad \tilde{q}_1 = \frac{\delta_1}{\mu_1\delta x}f + \frac{\delta_2\mu_2}{\delta y}g, \quad \tilde{q}_2 = \frac{\delta_1\mu_1}{\delta x}f + \frac{\delta_2}{\mu_2\delta y}g,$$

where μ_1 and μ_2 in denominators should be understood as in Pade fraction operators.

Following the same lines as in the proof of Theorem 2, it can be shown that a general scheme in form (25) is vorticity preserving for any choice of $\tilde{q}_1$ and $\tilde{q}_2$ if and only if $\Phi_1 = \Phi_2 = 0$, which excludes dissipative schemes. Thus the form (25) is not well-suited to the construction of vorticity-preserving schemes.

For the special choices defining Scheme (24), the discrete vorticity equation reduces to:

$$D_0\mu_0\tilde{\omega} = \tilde{\mathrm{RHS}}$$

with

$$\tilde{\mathrm{RHS}} = \frac{\Delta t}{2}D_1D_2(\tilde{q}_2^{(1)} - \tilde{q}_1^{(1)}) = \frac{\Delta t}{2}\frac{\delta_1}{\delta x}\frac{\delta_2}{\delta y}\left[\frac{\delta_2\mu_1}{\delta y}(1-\mu_2^2)v - \frac{\delta_1\mu_2}{\delta x}(1-\mu_1^2)u\right].$$

Since $\tilde{\mathrm{RHS}}$ is not null, Scheme (24) is not vorticity preserving. Numerical experiments confirm that vorticity is diffused by this scheme.

(c) Consider a residual-based scheme defined by

$$(\Lambda\tilde{r})^{n+1}_{j,k} = 0, \tag{26}$$

where $\tilde{r}$ denotes the discrete residual:

$$\tilde{r}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \left(\mu_1\mu_2\frac{\overline{\Delta}w}{\Delta t} + \frac{\delta_1\mu_2}{\delta x}f + \frac{\delta_2\mu_1}{\delta y}g\right)^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} \tag{27}$$

and Λ is the difference operator:

$$\Lambda = \mu_1\mu_2 - \frac{1}{2}(\delta_1\mu_2\Phi_1 + \delta_2\mu_1\Phi_2), \tag{28}$$

with the matrices:

$$\Phi_1 = \frac{\min(\delta x, \delta y)}{\delta x} A, \quad \Phi_2 = \frac{\min(\delta x, \delta y)}{\delta y} B. \tag{29}$$

This scheme is second order accurate (in time and space) and unconditionally stable in L_2. It can be cast in the form (12) with the above Φ_1, Φ_2 and

$$D_0 = \frac{1}{\Delta t}\overline{\Delta}, \quad D_1 = \frac{1}{\delta x}\delta_1\mu_2, \quad D_2 = \frac{1}{\delta y}\delta_2\mu_1,$$
$$\mu_0 = \mu^2, \quad \mu = \mu_1\mu_2,$$
$$h_1 = \frac{\delta x}{2}, \quad h_2 = \frac{\delta y}{2},$$
$$\tilde{q} = \tilde{r}.$$

Condition (15) is satisfied with $h = \min(h_1, h_2)$, so that scheme (26)–(29) is vorticity preserving, namely

$$\left(\mu_0 \frac{\overline{\Delta}\tilde{\omega}}{\Delta t}\right)^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = 0, \tag{30}$$

where $\tilde{\omega}$ is the compact vorticity (22). By defining the starting scheme (for $n = 0$) of the 3 time-level scheme (26)–(29) as the same scheme in which $\overline{\Delta}$ is simply replaced by Δ, we can deduce from (30):

$$(\mu_0\tilde{\omega})^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = (\mu_0\tilde{\omega})^{n}_{j+\frac{1}{2},k+\frac{1}{2}}. \tag{31}$$

(d) Let us point out that the residual-based schemes considered in [8] cannot be put in the form (12) for the same reason as the one given above for the Lax–Wendroff version (24). Again, numerical experiments show that vorticity is diffused by these schemes.
(e) Note that numerical dissipations based on higher-order derivatives can also be investigated using Theorem 2 by choosing $\tilde{q}$ properly.

3. Linearised Euler equations

Linearising the Euler equations around a uniform flow at speed (u_0, v_0) gives the equations of acoustics with advection. In a dimensionless form where the sound speed and the density of the uniform flow are equal to one, these equations can be written in the form (1) with

$$w = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad f = f(w) = \begin{bmatrix} pu_0 + u \\ uu_0 + p \\ vu_0 \end{bmatrix}, \quad g = g(w) = \begin{bmatrix} pv_0 + v \\ uv_0 \\ vv_0 + p \end{bmatrix} \tag{32}$$

and the same notations as in Section 2.

By taking the curl of the velocity equations, we obtain

$$\partial_t\omega + u_0\partial_x\omega + v_0\partial_y\omega = 0. \tag{33}$$

Here the vorticity is advected at the constant speed (u_0, v_0) without alteration. Similarly as in pure acoustics, there is no creation or diffusion of vorticity.

Consider again the general approximation (12) of System (1) and define the discrete vorticity by the expression (13).

Definition 1. For the linearised Euler equations, the scheme (12) is said to be vorticity preserving if the discrete vorticity $\tilde{\omega}$ satisfies

$$(D_0\mu_0 + u_0D_1\mu + v_0D_2\mu)\tilde{\omega} = 0. \tag{34}$$

This definition means that the numerical dissipation of Scheme (12), i.e. the right-hand side of (12), does not appear in the discrete vorticity equation. Note that for pure acoustics ($u_0 = v_0 = 0$), Eq. (34) reduces to (14), that is Definition 1 extends the one used previously.

For the linearised Euler equations, the Jacobian matrices of the flux are:

$$A = \begin{bmatrix} u_0 & 1 & 0 \\ 1 & u_0 & 0 \\ 0 & 0 & u_0 \end{bmatrix}, \quad B = \begin{bmatrix} v_0 & 0 & 1 \\ 0 & v_0 & 0 \\ 1 & 0 & v_0 \end{bmatrix}. \tag{35}$$

Let us denote by A_0 and B_0 the above matrices for $u_0 = v_0 = 0$, i.e. for pure acoustics.

Theorem 3. *For the linearised Euler equations, the scheme (12) is vorticity preserving for any choice of $\tilde{q}$ if and only if*

$$h_1\Phi_1 = hA_0, \quad h_2\Phi_2 = hB_0, \tag{36}$$

where h is a scalar parameter.

Proof 3. For the linearised Euler equations, the eigenvalues of A and B are:

$$a^{(1)} = u_0 - 1, \quad a^{(2)} = u_0, \quad a^{(3)} = u_0 + 1,$$
$$b^{(1)} = v_0 - 1, \quad b^{(2)} = v_0, \quad b^{(3)} = v_0 + 1,$$

and we have

$$A = T_A \operatorname{Diag}[a^{(i)}] T_A^{-1}, \quad B = T_B \operatorname{Diag}[b^{(i)}] T_B^{-1},$$

with the *same* matrices T_A and T_B as for pure acoustics. Therefore, the expressions of the matrices Φ_p in terms of their eigenvalues $\varphi_p^{(i)}$ are exactly the same as those given in Proof 1.

By taking the discrete curl of the velocity equations of the scheme (12), we obtain the discrete vorticity equation (16) with w, f and g now defined by (32).

The left-hand side of Eq. (16) becomes:

$$\tilde{\text{LHS}} = D_0\mu_0\tilde{\omega} + D_1\mu(u_0\tilde{\omega} - D_2 p) + D_2\mu(v_0\tilde{\omega} + D_1 p) = (D_0\mu_0 + u_0 D_1\mu + v_0 D_2\mu)\tilde{\omega}.$$

The right-hand side of Eq. (16) is still (17), so that the condition for $\tilde{\text{RHS}}$ to be zero for any $\tilde{q}$ is (18), which should now be written as the condition (36). □

Example: Consider the modified LWN scheme defined from the general scheme (12) as the LWN scheme (21) except that the matrices Φ_p are taken as:

$$\Phi_1 = A_0, \quad \Phi_2 = B_0. \tag{37}$$

Condition (36) is satisfied with $h = \Delta t/2$ so that this scheme is vorticity preserving for the linearised Euler equations. However, the difficulty is that the modification (37) corrupts the numerical stability as well as the second-order accuracy: the modified LWN scheme is first-order accurate and unstable, except when $u_0 = v_0 = 0$.

The above example shows that the assumption (36) may be too restrictive. Thus, let us now look for schemes of the general form (12) satisfying (15) – rather than (36) – that are vorticity preserving not for any choice of $\tilde{q}$, but for special choices of this $\tilde{q}$.

Lemma 1. *For the linearised Euler equations, the discrete vorticity equation of the scheme (12) with the condition (15) can be written as*

$$(D_0\mu_0 + u_0 D_1\mu + v_0 D_2\mu)\tilde{\omega} = h(u_0 D_1 + v_0 D_2)\tilde{curl}_q, \tag{38}$$

where

$$\tilde{curl}_q = D_1\tilde{q}^{(3)} - D_2\tilde{q}^{(2)}.$$

Proof. As shown before, the discrete curl of the velocity equations of the scheme (12) is

$$(D_0\mu_0 + u_0 D_1\mu + v_0 D_2\mu)\tilde{\omega} = \tilde{\text{RHS}},$$

where $\tilde{RHS}$ is given by (17). Owing to the condition (15) with the matrices (35), the eigenvalues of $h_1\Phi_1$ and $h_2\Phi_2$ are here equal to:

$$\bar{\varphi}_1^{(1)} = h(u_0 - 1), \quad \bar{\varphi}_1^{(2)} = hu_0, \quad \bar{\varphi}_1^{(3)} = h(u_0 + 1),$$
$$\bar{\varphi}_2^{(1)} = h(v_0 - 1), \quad \bar{\varphi}_2^{(2)} = hv_0, \quad \bar{\varphi}_2^{(3)} = h(v_0 + 1).$$

Thus, the right-hand side (17) reduces to:

$$\tilde{RHS} = h(u_0 D_1^2 \tilde{q}^{(3)} - u_0 D_1 D_2 \tilde{q}^{(2)} + v_0 D_1 D_2 \tilde{q}^{(3)} - v_0 D_2^2 \tilde{q}^{(2)}) = h(u_0 D_1 + v_0 D_2)\tilde{curl}_q$$

and the discrete vorticity equation becomes Eq. (38). □

Theorem 4. *For the linearised Euler equations, the scheme (12) with the condition (15) and*

$$\tilde{q} = D_1 f + D_2 g \tag{39}$$

is not vorticity preserving, except when $u_0 = v_0 = 0$.

Proof 4. For the choice (39), we have

$$\tilde{curl}_q = D_1(D_1 f^{(3)} - D_2 f^{(2)}) + D_2(D_1 g^{(3)} - D_2 g^{(2)}) = (u_0 D_1 + v_0 D_2)\tilde{\omega}.$$

Using Lemma 1, we obtain the discrete vorticity equation:

$$(D_0\mu_0 + u_0 D_1 \mu + v_0 D_2 \mu)\tilde{\omega} = h(u_0 D_1 + v_0 D_2)^2 \tilde{\omega}, \tag{40}$$

where

$$(u_0 D_1 + v_0 D_2)^2 = u_0^2 D_1^2 + 2u_0 v_0 D_1 D_2 + v_0^2 D_2^2.$$

Clearly, the right-hand side of Eq. (40) is not null in general, but produces a dissipation of the discrete vorticity. □

Example: The LWN scheme (21) is of the form (12) with the discrete operators D_0, D_1, D_2, μ_0 and μ defined in Example (a) of Section 2. It satisfies the condition (15) with $h = \Delta t/2$ and (39) holds. Therefore, the LWN scheme is not vorticity preserving for the linearised Euler equations, except for pure acoustics.

Contrary to the pure acoustics case, Theorem 4 shows that the choice of $\tilde{q} = D_1 f + D_2 g$ (as in the Lax–Wendroff–Ni scheme) *cannot ensure vorticity preserving* when advection occurs. Nevertheless the proof of Theorem 4 suggests that the difficulty can be tided over by adding the unsteady term $D_0\mu w$ to $\tilde{q}$, i.e. by taking $\tilde{q}$ as the full residual. This new option is considered in Theorems 5 and 6.

Theorem 5. *For the linearised Euler equations, the scheme (12) with the condition (15), $\mu_0 = \mu^2$ and*

$$\tilde{q} = D_0 \mu w + D_1 f + D_2 g \tag{41}$$

is vorticity preserving provided the discrete operator:

$$\Lambda_\omega = \mu - h(u_0 D_1 + v_0 D_2) \tag{42}$$

is non-singular.

Proof 5. For the choice (41), we have

$$\tilde{curl}_q = (D_0 \mu + u_0 D_1 + v_0 D_2)\tilde{\omega}$$

so that, from Lemma 1, the discrete vorticity equation reads

$$(D_0 \mu^2 + u_0 D_1 \mu + v_0 D_2 \mu)\tilde{\omega} = h(u_0 D_1 + v_0 D_2)(D_0 \mu + u_0 D_1 + v_0 D_2)\tilde{\omega}$$

which can also be written as:

$$\Lambda_\omega (D_0 \mu + u_0 D_1 + v_0 D_2)\tilde{\omega} = 0. \tag{43}$$

If Λ_ω is non-singular, this equation reduces to:

$$(D_0\mu + u_0 D_1 + v_0 D_2)\tilde{\omega} = 0$$

or (34) after applying μ, which shows the vorticity preservation. □

Example: The residual-based scheme (26)–(29) is of the form (12) with the discrete operator D_0, D_1, D_2 and μ given in Example (a) of Section 2 and $\mu_0 = \mu^2$. Since (15) and (41) holds, Theorem 5 shows that this scheme is vorticity preserving for the linearised Euler equations when Λ_ω is non singular. Here:

$$\Lambda_\omega = \mu_1\mu_2 - \frac{1}{2}\min(\delta x, \delta y)\left(u_0\frac{\delta_1\mu_2}{\delta x} + v_0\frac{\delta_2\mu_1}{\delta y}\right).$$

The Fourier symbol of Λ_ω is:

$$\hat{\Lambda}_\omega = \cos\frac{\xi}{2}\cos\frac{\eta}{2}\left[1 - \mathrm{i}\left(\alpha_1 u_0\tan\frac{\xi}{2} + \alpha_2 v_0 \tan\frac{\eta}{2}\right)\right], \tag{44}$$

where $\mathrm{i}^2 = -1$, ξ and η denote the reduced wave numbers in the x and y-directions and

$$\alpha_1 = \frac{\min(\delta x, \delta y)}{\delta x}, \quad \alpha_2 = \frac{\min(\delta x, \delta y)}{\delta y}.$$

The operator Λ_ω is non-singular if and only if the modulus of $|\hat{\Lambda}_\omega|$ does not vanish. This modulus is:

$$|\hat{\Lambda}_\omega|^2 = \left(\cos\frac{\xi}{2}\cos\frac{\eta}{2}\right)^2\left[1 + \left(\alpha_1 u_0\tan\frac{\xi}{2} + \alpha_2 v_0\tan\frac{\eta}{2}\right)^2\right].$$

Clearly $|\hat{\Lambda}_\omega|$ does not vanish except when

$$\cos\frac{\xi}{2} = 0 \quad \text{or} \quad \cos\frac{\eta}{2} = 0$$

that is

$$\xi = \pi \quad \text{or} \quad \eta = \pi \ (\mathrm{mod}\, 2\pi),$$

which corresponds to the shortest wavelengths.

In the above theorem, the scheme is supposed to satisfy the condition (15), i.e. to use dissipation matrices Φ_1 and Φ_2 proportional to A and B, which excludes more sophisticated dissipation matrices such as those encountered in upwind schemes. For the special choice (41) of $\tilde{q}$, it is possible to get rid of Condition (15) and in addition to remove the assumption on Φ_1 and Φ_2 eigenvectors made in the definition of Scheme (12). With the choice (41), Theorem 5 can be extended as follows.

Theorem 6. *For the linearised Euler equations, the scheme* (12) *with* $\mu_0 = \mu^2$ *and* $\tilde{q}$ *defined by* (41) *is vorticity preserving provided the discrete operator*:

$$\Lambda = \mu - h_1 D_1 \Phi_1 - h_2 D_2 \Phi_2 \tag{45}$$

is non-singular.

Proof 6. By taking the curl of the velocity equations of the scheme (12), we obtain the discrete vorticity equation:

$$(D_0\mu_0 + u_0 D_1\mu + v_0 D_2\mu)\tilde{\omega} = h_1 D_1 \widetilde{curl}_1 + h_2 D_2 \widetilde{curl}_2, \tag{46}$$

where

$$\widetilde{curl}_p = D_1(\Phi_p\tilde{q})^{(3)} - D_2(\Phi_p\tilde{q})^{(2)}, \quad p = 1, 2, \tag{47}$$

$$\tilde{q} = D_0\mu w + D_1 f + D_2 g. \tag{48}$$

Now for $\mu_0 = \mu^2$, the scheme (12) can be written as

$$\mu\tilde{q} = h_1 D_1(\Phi_1\tilde{q}) + h_2 D_2(\Phi_2\tilde{q})$$

that is

$$\Lambda\tilde{q} = 0$$

with Λ given by (45).

If Λ is non-singular, we obtain $\tilde{q} = 0$ so that the right-hand side of (46) vanishes and the scheme is vorticity preserving. □

4. Full Euler equations

The Euler equations are of the form (1) with

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \quad f(w) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (\rho E + p)u \end{bmatrix}, \quad g(w) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (\rho E + p)v \end{bmatrix}, \tag{49}$$

where ρ is the density, p the pressure, u and v the components of the fluid velocity $\vec{V}$ and E is the specific total energy. The system is closed by a thermodynamic state equation.

A common way to derive the vorticity equation consists in taking the curl of the velocity equation:

$$\partial_t\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} + \frac{1}{\rho}\vec{\nabla}p = 0.$$

This leads to the well-known equation:

$$\partial_t\vec{\omega} + (\vec{V}\cdot\vec{\nabla})\vec{\omega} = (\vec{\omega}\cdot\vec{\nabla})\vec{V} - (\vec{\nabla}\cdot\vec{V})\vec{\omega} + \frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p \tag{50}$$

for the vorticity vector $\vec{\omega} = \vec{\nabla}\times\vec{V}$.

The left-hand side of (50) is the advective derivative of $\vec{\omega}$. The right-hand side contains the terms modifying the ordinary vorticity transport: the first term represents the stretching and warping of the vorticity tubes (it vanishes in the 2-D case), the second term accounts for the compressibility effect (it vanishes when $\vec{\nabla}\cdot\vec{V} = 0$) and the last term describes the baroclinic effects (it vanishes in the barotropic case, e.g. for isentropic flows).

However, since the discrete form of the vorticity equation (50) is not easy to obtain for a conservative numerical scheme, we will rather use a vorticity equation derived by directly taking the curl of the momentum equations:

$$\partial_t(\rho\vec{V}) + \vec{\nabla}\cdot(\rho\vec{V}\otimes\vec{V}) + \vec{\nabla}p = 0.$$

This leads to

$$\partial_t\vec{\Omega} + \vec{\nabla}\times[\vec{\nabla}.(\rho\vec{V}\otimes\vec{V})] = 0, \tag{51}$$

with

$$\vec{\Omega} = \vec{\nabla}\times(\rho\vec{V}) = \rho\vec{\omega} + \vec{\nabla}\rho\times\vec{V}.$$

Remarks:

(i) The pressure is not involved in Eq. (51) in the general case.

(ii) The linearisation of Eq. (51) around a constant state of density $\rho_0 = 1$ and velocity $\vec{V}_0$ gives

$$(\partial_t + \vec{V}_0\cdot\vec{\nabla})\vec{\omega} + \vec{\nabla}e_\rho\times\vec{V}_0 = 0 \tag{52}$$

with

$$e_\rho = (\partial_t + \vec{V}_0 \cdot \vec{\nabla})\rho' + \vec{\nabla} \cdot \vec{V}',$$

where ρ' and $\vec{V}'$ are the density and velocity perturbations. Since the linearised mass equation is $e_\rho = 0$, Eq. (52) reduces to the vorticity equation (33) previously used for the linearised Euler equations.

In two space dimensions, the vector $\vec{\Omega}$ reduces to its component Ω normal to the x, y-plane:

$$\Omega = \partial_x(\rho v) - \partial_y(\rho u) \tag{53}$$

and the vorticity equation (51) reads

$$\partial_t\Omega + \partial_x[\partial_x(\rho uv) + \partial_y(\rho v^2)] - \partial_y[\partial_x(\rho u^2) + \partial_y(\rho uv)] = 0. \tag{54}$$

Be given a conservative scheme, obtaining a discrete form of Eq. (54) will simply consist in taking the discrete curl of the second and third components of the scheme.

Definition 2. For the Euler equations, the scheme (12) is said to be vorticity preserving if

$$D_0\mu_0\tilde{\Omega} + D_1[D_1\mu(\rho uv) + D_2\mu(\rho v^2)] - D_2[D_1\mu(\rho u^2) + D_2\mu(\rho uv)] = 0, \tag{55}$$

where

$$\tilde{\Omega} = D_1(\rho v) - D_2(\rho u). \tag{56}$$

We are now able to extend Theorem 6 to the full Euler equations.

Theorem 7. *For the Euler equations, the scheme (12) with $\mu_0 = \mu^2$ and $\tilde{q}$ defined by (41) is vorticity preserving provided the discrete operator Λ in (45) is non-singular.*

Proof 7. By taking the discrete curl of the momentum equations of the scheme (12), we obtain:

$$D_0\mu_0\tilde{curl}_w + D_1\mu\tilde{curl}_f + D_2\mu\tilde{curl}_g = h_1D_1\tilde{curl}_1 + h_2D_2\tilde{curl}_2 \tag{57}$$

with

$$\tilde{curl}_w = D_1(\rho v) - D_2(\rho u) = \tilde{\Omega},$$
$$\tilde{curl}_f = D_1(\rho uv) - D_2(\rho u^2 + p), \quad \tilde{curl}_g = D_1(\rho v^2 + p) - D_2(\rho uv),$$

and (47), (48). Clearly, the pressure disappears in the left-hand side of Eq. (57). With the same argument as in Proof 6, the right-hand side vanishes and Eq. (57) reduces to the definition Eq. (55). □

Example: The residual-based scheme (26)–(28) with any dissipation matrices Φ_1 and Φ_2 is of the form (12) with $\mu_0 = \mu^2$ and it satisfies (41). Theorem 6 shows that this residual-based scheme is vorticity preserving for the Euler equations provided the operator Λ defined by (45) is non-singular. For instance, matrices Φ_1 and Φ_2 proposed in [8] are acceptable, except for the shortest wavelengths of the numerical solution.

5. Numerical validations for the Euler equations

5.1. Numerical schemes

5.1.1. RBV2 scheme

In this section, the residual-based scheme (26)–(28) with the second-order time differencing formula (20) will be applied to the full Euler equations for computing steady and unsteady vortex flows. The dissipation matrices used are defined as:

$$\Phi_1 = T_A\,\mathrm{Diag}[\varphi_1^{(i)}]T_A^{-1}, \quad \Phi_2 = T_B\,\mathrm{Diag}[\varphi_2^{(i)}]T_B^{-1}, \tag{58}$$

where

$$\varphi_1^{(i)} = \mathrm{sgn}(a^{(i)})\varphi^{(i)}, \quad \varphi_2^{(i)} = \mathrm{sgn}(b^{(i)})\psi^{(i)},$$
$$\varphi^{(i)} = \min\left(1, \frac{\delta y|a^{(i)}|}{\delta x \max_l |b^{(l)}|}\right), \quad \psi^{(i)} = \min\left(1, \frac{\delta x|b^{(i)}|}{\delta y \max_l |a^{(l)}|}\right).$$

These dissipation matrices are close to those of [8], except the replacement of $\min_l$ by $\max_l$ in $\varphi^{(i)}$ and $\psi^{(i)}$ to avoid difficulties with zero eigenvalues when there is no advection. Although the schemes in [8] are residual-based, let us recall that they cannot be cast in the form (12) because they do not use a single $\tilde{q}$ in their numerical dissipation. Experiments with these schemes clearly demonstrate vorticity diffusion. On the contrary, the residual-based scheme (26)–(28) is vorticity preserving. Since it is also second-order accurate in time and space, we will refer to it as the RBV2 scheme. This scheme is fully implicit in time and unconditionally stable. It is solved iteratively using a dual time technique. A study of the dissipation properties shows that for diagonal flows, the shortest wavelengths are dissipated in the streamwise direction but not in the cross direction. This is sufficient to perform the present test cases but for the computation of more complex flows, high-order filters [4] might be necessary to avoid spurious oscillations in the cross directions. Concerning the precision of RBV2, a theoretical analysis of the modified equation in the case of the linearised Euler equations shows that for advection dominated flows at speed $\vec{V}_0$, the best solution is obtained on a square mesh ($\delta x = \delta y = h$) when using a time step satisfying:

$$\Delta t = \frac{1}{2}\frac{h}{|\vec{V}_0|}. \tag{59}$$

5.1.2. D3 scheme

To allow a clear appreciation of the vorticity-preservation property, the numerical results of the RBV2 scheme will be compared to those given by a directional third-order scheme [12], called D3. This scheme is similar to the well-known Roe-MUSCL scheme (third-order version without limiter or entropy correction [13]). The D3 scheme reads:

$$\left(\frac{3w^{n+1} - 4w^n + w^{n-1}}{2\Delta t}\right)_{j,k} + \left(\frac{\delta_1 F}{\delta x} + \frac{\delta_2 G}{\delta y}\right)_{j,k}^{n+1} = 0 \tag{60}$$

with the numerical fluxes:

$$\begin{aligned} F_{j+\frac{1}{2},k} &= \left[\left(\mathrm{I} - \frac{\delta_1^2}{6}\right)\mu_1 f + \frac{1}{12}|A_\mathrm{R}|\delta_1^3 w\right]_{j+\frac{1}{2},k}, \\ G_{j,k+\frac{1}{2}} &= \left[\left(\mathrm{I} - \frac{\delta_2^2}{6}\right)\mu_2 g + \frac{1}{12}|B_\mathrm{R}|\delta_2^3 w\right]_{j,k+\frac{1}{2}}, \end{aligned} \tag{61}$$

where A_R and B_R are Roe's averages [14] of the Jacobian flux matrices at the interfaces. This scheme is fully implicit in time and unconditionally stable. It is solved with a dual time technique similarly as the RBV2 scheme.

5.2. Steady vortex

We consider the 2-D inviscid vortex proposed by Yee et al. [7], for which the entropy is uniform. The vortex is initially located at the origin ($x = y = 0$). Its velocity components u, v and the absolute temperature T are defined in non-dimensional form as:

$$u = -\frac{\Gamma}{2\pi} y \exp\left(\frac{1-r^2}{2}\right), \quad v = \frac{\Gamma}{2\pi} x \exp\left(\frac{1-r^2}{2}\right), \quad T = 1 - \frac{(\gamma-1)\Gamma^2}{8\gamma\pi^2}\exp(1-r^2), \tag{62}$$

where $r^2 = x^2 + y^2$. The vortex strength Γ is set equal to 5. The thermodynamic equation of state is the ideal law $p = \rho T$, with constant specific heats of ratio $\gamma = 1.4$. The uniformity of entropy gives $\rho = T^{1/\gamma-1}$. This vortex is a steady solution of the Euler equations. Numerically, we solve the unsteady Euler equations using the above vortex as the initial condition. The observed evolution is due uniquely to the numerical errors, namely the dissipative errors that are not compensated by any physical effect, contrary to the case of a shock wave where nonlinear compression can balance dissipation. The computational domain extends from −5 to 5 in the x and y-directions. Periodic boundary conditions are applied in both directions. A regular Cartesian grid is used with 50×50 cells ($\delta x = \delta y = 0.2$). The evolution is computed up to time $t_f = 500$ with a time step $\Delta t = 1$, i.e. $\mathrm{CFL} = \frac{\Delta t}{\delta x}\max(\sqrt{u^2+v^2} + \sqrt{\gamma T}) = 9.6$. The vorticity fields shown in Fig. 1 and the cut of the solutions shown in Fig. 2 clearly assess that the stationary vortex (62) is seen as a steady solution by the RBV2 scheme while it is not the case for the D3 scheme. After 500 iterations the initial vortex is indeed mainly dissipated by the D3 scheme. The numerical dissipation of D3 is also responsible for the symmetry-loss of the computed solution.

5.3. Vortex advection along x-axis

We consider now the advection of the same vortex at constant velocity $\vec{V}_0$ of components $u_0 = 0.5$, $v_0 = 0$. The initial condition is the same as in the steady vortex case except that the velocity components (62) are now augmented with the velocity (u_0, v_0). This unsteady problem is solved in the same computational domain, with

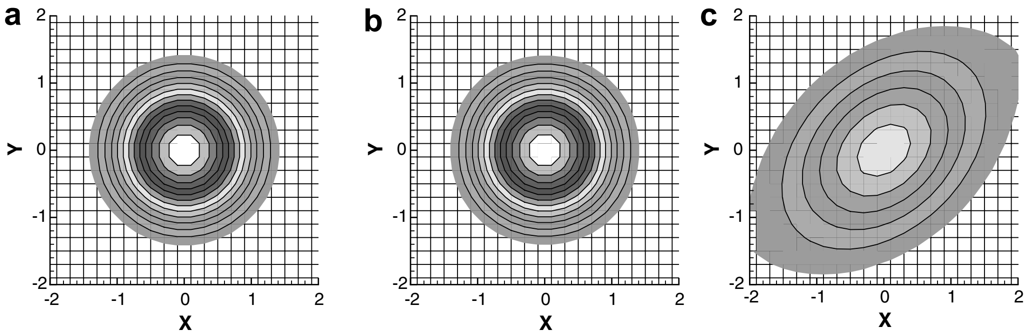

Fig. 1. Evolution of a steady vortex. Vorticity fields of the (a) exact, (b) RBV2 scheme and (c) D3 scheme solutions at $t = 500$, with $\delta x = \delta y = 0.2$.

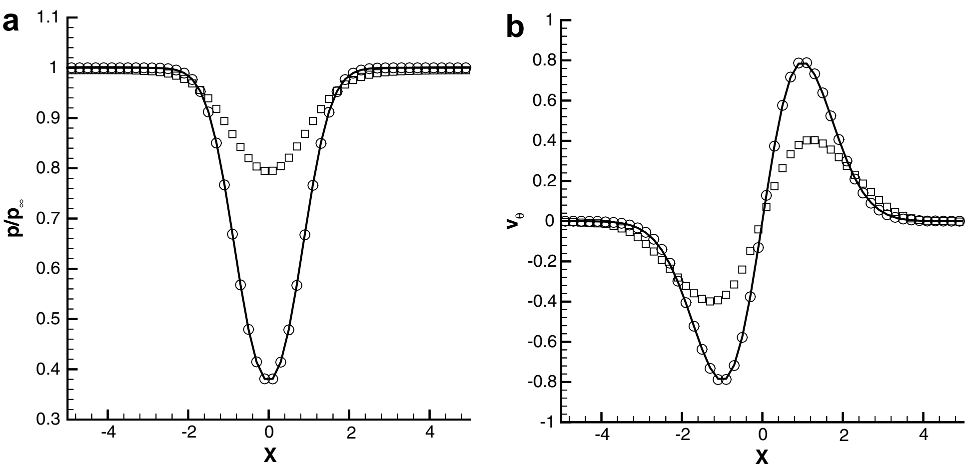

Fig. 2. Evolution of a steady vortex. Cuts of the exact (–), RBV2 scheme (○) and D3 scheme (□) solutions at the centre-line $y = 0$, at $t = 500$, with $\delta x = \delta y = 0.2$: (a) pressure, (b) tangential velocity.

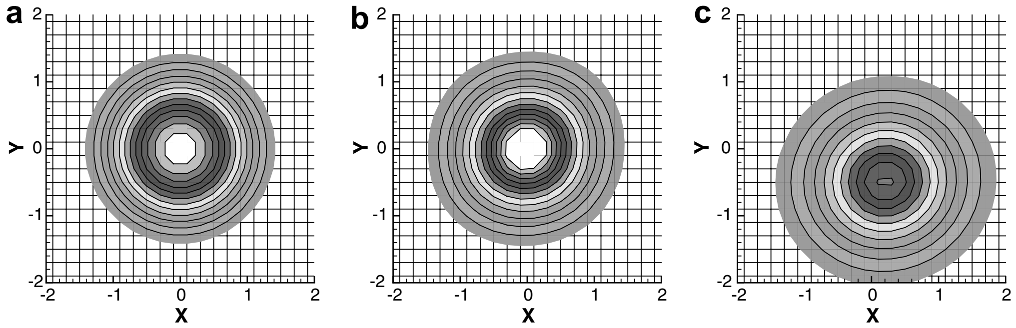

Fig. 3. Advection of a vortex. Vorticity fields of the (a) exact, (b) RBV2 scheme and (c) D3 scheme solutions at $t = 100$, with $\delta x = \delta y = 0.2$.

the same periodic boundary conditions and the same grid as in the steady case. The time step is $\Delta t = 0.2$, i.e. CFL = 2.42, which satisfies (59). The vortex evolution is computed up to time $t_f = 100$, corresponding to 5 crossings of the domain by the vortex. The vorticity fields are displayed in Fig. 3 and the solutions along a x-line passing through the vortex centre is shown in Fig. 4. Contrary to the steady-case, the time discretisation now plays a role in the solution. With the optimal choice of the time step (59), the vortex calculated by RBV2 is more precisely located than the one calculated by D3. But, the important point is that the spatial numerical dissipation of RBV2 scheme vanishes in the discrete vorticity equation, which allows a better representation of the vortex solution.

5.4. Oblique vortex advection

Again we keep the same vortex and consider its advection at velocity modulus $|\vec{V}_0| = 0.5$ in various directions, of angles $\alpha = 0$, $\pi/8$ and $\pi/4$ with respect to the x-axis. Owing to the symmetries in the RBV2 scheme, the results obtained are representative of an advection problem along space direction varying from $\alpha = 0$ to $\alpha = 2\pi$ with step angle $\delta\alpha = \pi/8$. The evolution is computed up to time $t_f = 100$ in the same conditions as above. Fig. 5 presents the vorticity fields at different times. Snapshots are shown at $t = 0$, 20, 40, 60, 80 and 100 for the three advection directions considered. This visualisation shows that the RBV2 scheme produces much accurate vortex solutions that depend very weakly on the advection direction.

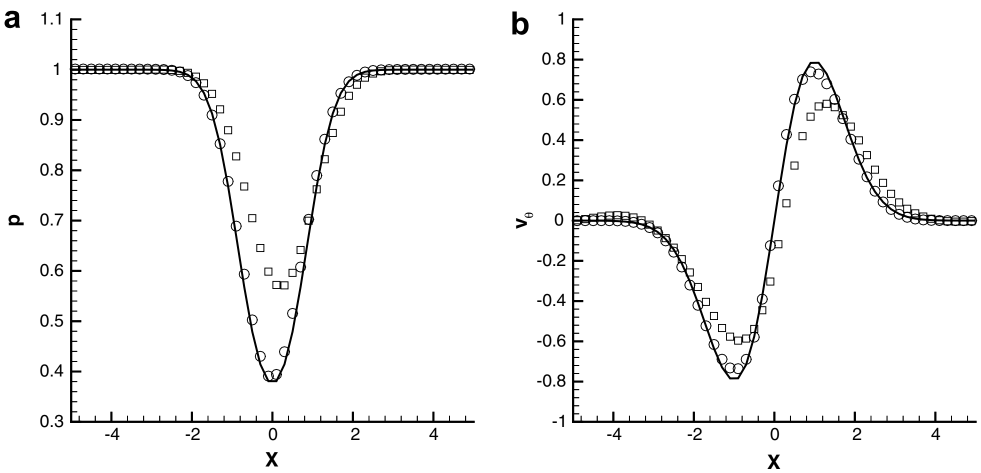

Fig. 4. Advection of a vortex. Cuts of the exact (–), RBV2 scheme (○) and D3 scheme (□) solutions at the centre of the vortex, at $t = 100$, with $\delta x = \delta y = 0.2$: (a) pressure, (b) tangential velocity.

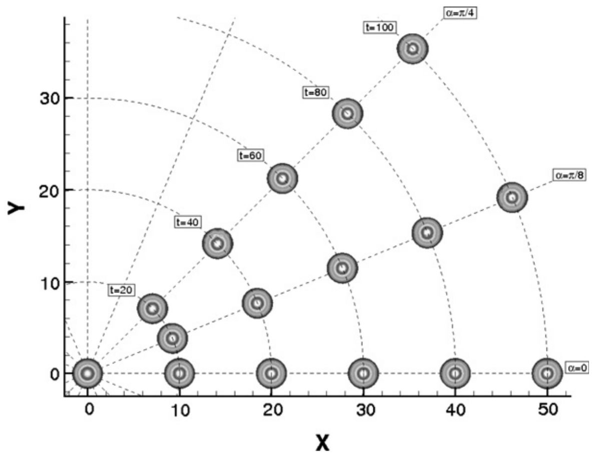

Fig. 5. Vorticity fields computed by the RBV2 scheme for several advection directions. Solutions at $t = 0$, 20, 40, 80 and 100, with $\delta x = \delta y = 0.2$.

6. Note on the extension to the 3-D Euler equations

Consider the three-dimensional Euler equations written in a Cartesian coordinate system (x_1, x_2, x_3) as

$$\partial_t w + \sum_i \partial_i f_i = 0, \tag{63}$$

where ∂_i denotes the spatial derivative $\partial/\partial x_i$, $f_i = f_i(w)$ is a flux component and the summation is for $i = 1$, 2, 3.

These equations are approximated on a Cartesian mesh by the conservative scheme:

$$D_0\mu_0 w + \sum_i D_i \mu f_i = \sum_i h_i D_i(\Phi_i \tilde{q}), \tag{64}$$

where D_0 (resp. D_i) is a difference operator consistent with ∂_t (resp. ∂_i), μ_0 and μ are discrete spatial operators consistent with the identity, h_i is a positive parameter depending on powers of the discretisation steps and tending to zero with them. All the space operators (D_1, D_2, D_3, μ and μ_0) are centred. The right-hand side of (64) is supposed to be a numerical dissipation in which Φ_1, Φ_2 and Φ_3 are matrices and $\tilde{q}$ is a vector depending on discrete functions of w, f_1, f_2 and f_3.

The discrete vorticity $\tilde{\Omega}$ is defined as a discrete curl of the momentum $\rho\vec{V}$, i.e.

$$\tilde{\Omega} = \tilde{\nabla} \times (\rho\vec{V})$$

with the discrete operator

$$\tilde{\nabla} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}.$$

Denoting by u_i the Cartesian components of the velocity $\vec{V}$, the Cartesian components of $\tilde{\Omega}$ can be written as:

$$\tilde{\Omega}_1 = D_2(\rho u_3) - D_3(\rho u_2),$$
$$\tilde{\Omega}_2 = D_3(\rho u_1) - D_1(\rho u_3),$$
$$\tilde{\Omega}_3 = D_1(\rho u_2) - D_2(\rho u_1).$$

With the $\tilde{\nabla}$ operator, the momentum equations of Scheme (64) read:

$$D_0\mu_0(\rho\vec{V}) + \tilde{\nabla} \cdot \mu(\rho\vec{V} \otimes \vec{V}) + \tilde{\nabla}\mu p = d,$$

where d is the numerical dissipation.

Considering the exact vorticity equation (51), Definition 2 should be generalised as follows.

Definition 3. For the Euler equations, the scheme (64) is said to be vorticity preserving if

$$D_0\mu_0\tilde{\Omega} + \tilde{\nabla} \times [\tilde{\nabla} \cdot \mu(\rho\vec{V} \otimes \vec{V})] = 0. \quad (65)$$

It is now straightforward to extend the proof of Theorem 7 and get the following main result.

Theorem 8. *For the Euler equations, the scheme (64) with $\mu_0 = \mu^2$ and*

$$\tilde{q} = D_0\mu w + \sum_i D_i f_i \quad (66)$$

is vorticity preserving provided the discrete operator

$$\Lambda = \mu - \sum_i h_i D_i \Phi_i$$

is non-singular.

7. Conclusion

A simple vorticity criterion has been derived for a general dissipative scheme applied to the acoustic system (Theorem 2). This criterion allows to recover the result found by Morton and Roe about the Lax–Wendroff–Ni scheme, but also to identify a residual-based scheme that preserves the vorticity and give a tool for studying higher order schemes.

When extending the concept to the Euler equations, vorticity preserving no longer means that vorticity is time invariant, but that the discrete vorticity equation is properly not affected by the numerical dissipation present in the scheme. From Theorem 4, the Lax–Wendroff–Ni scheme is proved to be not vorticity preserving for the linearised Euler equations, except for pure acoustics. On the contrary, schemes involving a time derivative in the numerical dissipation as the residual-based scheme can be satisfactory for the linearised and full Euler equations (Theorems 6–8). Vortex calculations with the residual-based scheme RBV2 confirm the interest of a vorticity-preserving scheme, specially when the main flow advects the vortex in an oblique direction. Extension of the RBV2 scheme to curvilinear meshes and application to a blade–vortex interaction problem will be presented in a forthcoming paper.

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